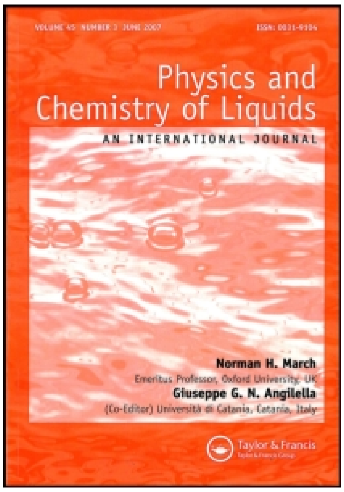

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THE DEFORMATION OF BUBBLES AND DROPS IN IMMISCIBLE FLUIDS

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We consider the uniform motion of a drop of one fluid moving in another immiscible, unbounded, fluid. We examine the condition of normal stress balance across the interface. Consistent application of this condition to linear order in the velocity requires the introduction of a new parameter, Δ, which takes into account deformations of the drop from a spherical shape. Imposing a condition for the critical size of these deformations allows us to predict the maximum size of the drop for a given terminal velocity. For the case of raindrops this gives a maximum size of approximately 2 millimeters.

KEY WORDS: Stress balance, critical size, rain drops.

The motion of bubbles in liquids is of considerable interest in problems of engineering. The subject was already investigated from the days of Stokes[1]. Interest in the subject is still very active and many papers appear on the shape oscillations of drops and bubbles. It is generally assumed that the deviations from the spherical shape are of an ellipsoidal form[2]. We find this assumption questionable since there is no reason to expect a symmetry about a plane through the center of the drop perpendicular to the direction of motion.

We have re-examined the supposedly well understood problem of the steady state motion of a bubble. We find that the boundary conditions on the solution of the Navier-Stokes equation are not applied systematically up to first order in the Reynolds number. Consider a bubble or a drop of radius "a" immersed entirely in another immiscible fluid unbounded outside. Let the steady state velocity of the drop be denoted by q. Taking spherical polar coordinates r, θ, ϕ with the origin at the center of the drop and the z axis along the direction of q, we have complete symmetry about ϕ. In such a coordinate system the center of the drop will be stationary, and the fluids inside and outside the drop will develop velocity fields $\hat{\mathbf{v}}'$ and $\mathbf{v}'$ and pressure fields $\hat{p}'$ and p'. We denote quantities inside the drop or bubble with a caret and quantities outside are without a caret. The velocity and pressure fields are determined from a linearised, time-independent form of the Navier-Stokes equations. It is assumed that the velocity is small enough to allow linearization and that the steady state condition allows for time independence. For incompressible fluids the continuity equation gives

$$\nabla \cdot \mathbf{v} = 0 \tag{1}$$

Equation (1) enables us to solve for v_θ if v_r is known. Following the method similar to that given in Sommerfeld[1], one obtains

$$p' = -\frac{A}{r^2}\cos\theta \tag{2}$$

$$\hat{p}' = \hat{A} r \cos\theta \tag{3}$$

$$v'_r = \left(-\frac{A}{\mu r} + B + \frac{C}{r^3}\right) \tag{4}$$

$$\hat{v}'_r = \left(-\frac{\hat{A}}{10\hat{\mu}} + \hat{B} + \frac{\hat{C}}{r^3}\right) \tag{5}$$

Here μ and $\hat{\mu}$ are the viscosities of the fluids $A, B, C, \hat{A}, \hat{B}, \hat{C}$ are to be determined by applying the boundary conditions. $\hat{v}'_\theta$ and v'_θ are found using Eq. (1). There are six unknown parameters in Eqs. (2–5). A seventh parameter comes from $\hat{p}_0 - p_0$, the difference between the static pressures inside and outside the bubble. The seven unknowns are determined from the seven boundary conditions discussed immediately below.

Asymptotic values of the velocity components in the outside fluid must give a uniform velocity in the z direction,

$$v'_r|_{r\to\infty} = q\cos\theta \tag{6}$$

$$v'_\theta|_{r\to\infty} = -q\sin\theta. \tag{7}$$

Immiscibility of the fluids yields

$$v'_r|_{r=a} = 0 \tag{8}$$

$$\hat{v}'_r|_{r=a} = 0. \tag{9}$$

Continuity of the tangential stress gives leads to

$$\mu r \frac{\partial}{\partial r} v'_\theta\bigg|_{r=a} = \hat{\mu} r \frac{\partial}{\partial r} \hat{v}'_\theta\bigg|_{r=a}, \tag{10}$$

and continuity of the tangential velocity across the interface gives

$$v'_\theta|_{r=a} = \hat{v}'_\theta|_{r=a}. \tag{11}$$

These six boundary conditions are adequate to determine six unknowns $A, B, C, \hat{A}, \hat{B}, \hat{C}$. The last, Eq. (11) is often called the slip condition. Actually we consider this a misnomer. A real slip should correspond to a discontinuity in the tangential velocities across the interface[3]. There is no compelling physical principle upon which this boundary condition is based, but it seems to phenomenologically realised. We harbour, however, some reservations about its strict application. The interface of two immiscible flowing fluids is a complicated system which has not been thoroughly investigated.

There is one more boundary condition, namely the continuity of the normal stress across the interface. This condition has not been systematically applied. It has only been used in the limit $q = 0$. In this limit one obtains

$$\left.\frac{2T}{r}\right|_{r=a} = \hat{p}_0 - p_0. \tag{12}$$

where T is the surface tension between the two fluids. A velocity profile obtained from Eqs. (4) and (5) using boundary conditions (6)–(11) is qualitatively sketched in Figure 1.

The drag force which is a frictional force, is given by[4]

$$F_{\text{drag}} = 2\pi\mu q a\left(\frac{3\hat{\mu} + 2\mu}{\hat{\mu} + \mu}\right). \tag{13}$$

q is determined self-consistently by imposing

$$F_{\text{drag}} = -F_{\text{gravity}} = \frac{4\pi a^3}{3}(\rho - \hat{\rho})g. \tag{14}$$

This gives a terminal velocity

$$q = \frac{2a^2}{3\mu}(\rho - \hat{\rho})g\left(\frac{\hat{\mu} + \mu}{3\hat{\mu} + 2\mu}\right) \tag{15}$$

For a falling drop $\rho - \hat{\rho} < 0$ while for a rising bubble $\rho - \hat{\rho} > 0$.

Use of the boundary conditions, Eqs. (6–11), determine the velocity fields of the solution completely. The last condition which should be satisfied, namely that the

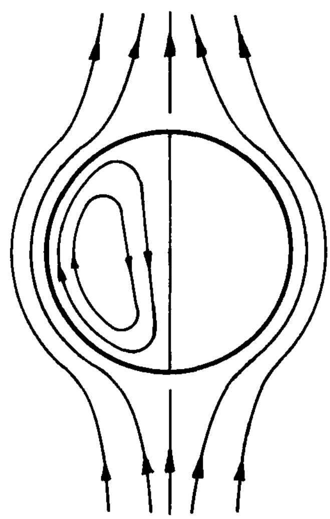

Figure 1 Qualitative velocity fields inside and outside a drop moving with constant velocity.

normal stress across the boundary must balance, has been used to compute the pressure inside the drop. In previous analyses it has, however, only been applied to the q independent, static stress across the interface. We find this consideration somewhat arbitrary. The correct condition of normal stress balance includes contributions from the viscous pressure terms, which are of course linear in q. The full condition is

$$\begin{aligned}(\sigma_{\text{surf.tens.}})_{rr} &= \hat{P}_{rr} - P_{rr} \\ &= (\hat{p}_0 + \hat{p}_1 - 2\hat{\mu}\hat{\dot{\varepsilon}}_{rr}) - (p_0 + p_1 - 2\mu\dot{\varepsilon}_{rr}) \qquad (16)\end{aligned}$$

where, inside and outside the bubble respectively, $\hat{P}_{rr}$ and P_{rr} are the rr components of the full pressure tensor, $\hat{p}_0$ and p_0 are the static q independent hydrostatic pres-sures (which have been considered in previous analyses), $\hat{p}_1$ and p_1 are the induced hydrodynamical pressures which are linear in q, $\hat{\dot{\varepsilon}}_{rr}$ and $\dot{\varepsilon}_{rr}$ correspond to the viscous stress given by

$$\dot{\varepsilon}_{rr} = \frac{\partial v_r}{\partial r} \qquad (17)$$

and $(\sigma_{\text{surf.tens.}})_{rr}$ is the normal stress due to the surface tension.

The continuity of normal stress in lowest order ($q = 0$) gives Eq. (13). The application of the full version of boundary condition, equation (16), gives inconsistent results since terms linear in q become equated to zero; the conditions Eqs. (6 – 11), determine all the velocities uniquely. Application of the normal stress condition up to first order in q over determines the system. One can imagine using the normal stress balance condition as a boundary condition ignoring one of the others for example, the continuity of tangential velocity, (11). This procedure does not seem to lead to sensible results. We must, therefore, introduce one additional parameter. We propose to allow the shape of the bubble to alter, and we use the normal stress condition up to first order in q to determine the deformation.

The hydrodynamical pressure and viscous stress depend non-trivially on the polar angle. Indeed

$$\hat{p}_1 = \hat{A} r \cos\theta \qquad (18a)$$

$$p_1 = \frac{A}{r^2}\cos\theta \qquad (18b)$$

while

$$\hat{\dot{\varepsilon}}_{rr} = \frac{2\hat{A}}{10\hat{\mu}} r\cos\theta \qquad (19a)$$

$$\dot{\varepsilon}_{rr} = \left(\frac{-A}{\mu}\left(\frac{1}{r^2} - \frac{3a^2}{r^4}\right) + \left(\frac{3qa^3}{r^4}\right)\right)\cos\theta \qquad (19b)$$

hence Eq. (16) is actually inconsistent for a spherical bubble. Previous analyses have simply neglected the hydrodynamic and viscous pressure terms.

We suggest the following minimal modification to remove this inconsistency. Physically, any moving bubble will be slightly distorted from a spherical shape, with the distortion proportional to q. All other fields will also be affected by this distortion but only at higher order. Taking into account these distortions in the normal stress balance equation leads to a consistent solution. We hypothesize a deformation given by

$$r(\theta) = a + \Delta cos\theta + o(\Delta^2) \tag{20}$$

where the new parameter Δ characterizes the induced pressure deformation. The normal stress due to the surface tension for such a deformed surface is given by

$$\frac{2T}{a + \Delta cos\theta} = \frac{2T}{a} - \frac{2T}{a^2}\Delta cos\theta + \cdots \tag{21}$$

Hence, from Eq. (16) we get

$$\frac{2T}{a} - \frac{2T}{a^2}\Delta cos\theta = \hat{p}_0 - p_0 + \left(\hat{A}a - \frac{A}{a^2} + 2\mu\left(\frac{2A}{\mu a^2} + \left(\frac{3q}{a}\right)\right) - 2\hat{\mu}\frac{2\hat{A}a}{10\hat{\mu}}\right)\cos\theta \tag{22}$$

This implies

$$\Delta = \left(\hat{A}a - \frac{A}{a^2} + 2\mu\left(\frac{2A}{\mu a^2} + \left(\frac{3q}{a}\right)\right) - 2\hat{\mu}\frac{2\hat{A}a}{10\hat{\mu}}\right) = \frac{-3aq\mu}{4T}\left(\frac{3\hat{\mu} + 2\mu}{\hat{\mu} + \mu}\right). \tag{23}$$

The fractional deformation $M = \Delta/a$ is then given by

$$M = \frac{a^2 g|(\rho - \hat{\rho})|}{2T} \tag{24}$$

replacing for q with the terminal velocity.

Equation (24) provides a relation for the fractional distortion of a bubble in terms of the radius of the bubble at the terminal velocity. It is seen that M is proportional to a^2. We observe that all drops suffer some deformation, but for larger drops the deformation can become of order unity. When $M \sim 1$, the deformation is so large that the analysis given will have to be modified. A change in the shape will always produce changes in the drag force. For a spherical bubble the drag force is linear in q. A change in the shape will produce terms of order q^3. The present calculation is correct up to terms linear in q and a separate detailed calculation will be necessary to investigate changes in the drag force due to changes in the shape. The drag force will change and the drop will break up at some critical $M = M_c < 1$. Thus the largest the drop will have a radius

$$a = \left(M_c\frac{2T}{|\rho - \hat{\rho}|g}\right)^{1/2} \tag{25}$$

Substituting for the parameters for a rain drop and taking $M_c = 1$ we find the largest raindrop will have a radius of 2 mm at terminal velocity. This elementary prediction seems consistent with every day observations.

It is often diffcult to make measurements under gravity when drops have large terminal velocities. If experiments are performed, however, with fluids of comparable, densities, the terminal velocities are not large. Detailed experimental observation of large drops or bubbles in fluids of comparable densities will enable us to get a precise value of M_c which is independent of the fluids considered.

Acknowledgements

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